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Solar Heating of a Rotating Cylinder with a Conduction Discontinuity

ROBERT J. EBY* AND ROBERT D. KARAM†
Fairchild Hiller Corporation,
Space and Electronics Systems Division,
Germantown, Md.

Nomenclature

A,B,C = dimensionless parameters
c = specific heat
F,f = solar input functions
G_1,G_2,G_3 = transform functions defined by Eqs. (13, 14, and 20)
$G(\theta_s)$ = function defined by Eq. (24)
k = thermal conductivity
r = radius
S = solar flux
t = thickness
T = absolute temperature
α = solar absorptance
ϵ = emittance
η = temperature fluctuation function
θ = angular position on circumference
ρ = density
σ = Stefan-Boltzmann constant
τ = dimensionless temperature defined by Eq. (4)
ψ = rotational position
ω = angular speed

Subscripts

m = mean
n = 1,2...
s = position of solar vector

Introduction

THE temperature distribution in rotating space vehicles in a solar environment is a subject of considerable engineering importance, and much attention has been given to the corresponding mathematical problems. Previous work on these problems[1,2] has been largely limited to systems in which conduction heat transfer is continuous throughout the solid portion of the body. During a uniform quasi-steady-state rotation of a sphere or a cylinder, the explicit dependence on time may be eliminated by relating time to spatial coordinates.[3] The mathematical model is then generated based on an established, nonchanging temperature profile as observed from a fixed frame of reference.

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* Manager, Thermal Control.
† Principal Engineer, Thermal Control.

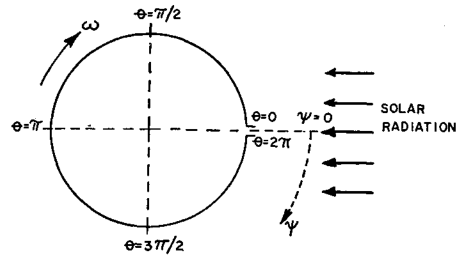

Fig. 1 Coordinate system.

This procedure cannot in general be implemented when the system involves a discontinuity in heat conduction; the temperature distribution will contain the explicit dependence on time as well as the spatial coordinates. In this Note, a thin cylindrical shell with a line conduction discontinuity is considered. The cylinder is rotating at a constant angular speed about the geometric axis which is normal to solar radiation. The temperature distribution is obtained as a function of time and dimensionless groups which include the thermal and geometric parameters. Comparison is made with a similar system which does not include the effects of discontinuous conduction. One direct application to this solution is in the evaluation of thermal bending of tubular extendible elements on spinning spacecrafts. Current designs of these elements incoporate seams of interlocking tabs along which conduction heat transfer is negligible.

Analysis

Consider an infinitely long thin cylindrical shell of open cross-section rotating uniformly in a solar radiation field as shown in Fig. 1. If it is assumed that 1) the open gap is small and its position with respect to the solar vector does not vary along the length, 2) internal radiation and heat interchange across the gap are negligible, 3) the thermophysical properties are independent of temperature and geometry, and 4) thermal distortions are small, the differential equation which relates heat conduction to the energy received, reradiated and stored may be written

$$\frac{\partial^2 T}{\partial \theta^2} + \frac{\alpha S r^2}{kt} f(\theta,\psi) - \frac{\epsilon r^2}{kt}\sigma T^4 = \frac{r^2 \rho c \omega}{k}\frac{\partial T}{\partial \psi} \quad (1)$$

with the boundary conditions (assumption 2)

$$\partial T/\partial\theta)_{\theta=0} = 0 \text{ and } \partial T/\partial\theta)_{\theta=2\pi} = 0 \quad (2)$$

Since only quasi-steady-state conditions are considered, the initial temperature may be taken as zero.

The input function $f(\theta,\psi)$, which characterizes the rotation of each point θ on the periphery with respect to solar radiation, is a rectified cosine wave which is displaced along the time axis by the magnitude of the angle, with $\theta = 0$ and $\theta = 2\pi$ defining the edges of the gap. If a delayed function of the form

$$F(\psi - \theta)\, U(\psi - \theta)$$

where $F(\psi) = f(0,\psi)$ and $U(\psi - \theta)$ is 1.0 for $\psi \geq \theta$ and 0 for $\psi < \theta$, is inserted into Eq. (1), then the required quasi-steady state temperature is obtained.

Equation (1) is linearized by introducing the following definitions[1]:

$$T = T_m(1+\eta) \qquad |\eta| \ll 1.0 \quad (3)$$

$$\tau = \eta + \tfrac{1}{4} \quad (4)$$

where η and τ are dimensionless parameters and T_m is a mean temperature which may be defined by

$$\epsilon\sigma T_m{}^4 = \frac{1}{2\pi}\int_0^{2\pi} \alpha S f(\theta,\psi_0) d\theta$$

where ψ_0 is the rotational angle at any given instant; or

$$T_m = (\alpha S/\pi\epsilon\sigma)^{1/4} \quad (5)$$

Using Eqs. (3) and (4), one can write

$$T = T_m(\tau + \tfrac{3}{4}) \quad (6)$$

and

$$T^4 \approx 4T_m{}^4\tau \quad (7)$$

The linearized energy equation becomes

$$\partial^2\tau/\partial\theta^2 + AF(\psi - \theta)U(\psi - \theta) - B\tau = C\partial\tau/\partial\psi \quad (8)$$

where $A \equiv \alpha Sr^2/ktT_m$, $B \equiv 4\epsilon r^2\sigma T_m{}^3/kt$, $C \equiv \rho c\omega r^2/k$. The Laplace transform of Eq. (8) with respect to ψ and with zero initial condition is

$$d^2\bar{\tau}/d\theta^2 + Ae^{-s\theta}\bar{F}(s) - B\bar{\tau} = Cs\bar{\tau} \quad (9)$$

with boundary conditions

$$d\bar{\tau}/d\theta)_{\theta=0} = 0 \text{ and } d\bar{\tau}/d\theta)_{\theta=2\pi} = 0 \quad (10)$$

Here, s is the Laplace variable and $\bar{F}(s)$ is given by

$$\bar{F}(s) = [1/(s^2+1)][s + 1/(2\sinh\pi s/2)] \quad (11)$$

The solution of Eq. (9) with boundary conditions (10) is

$$\bar{\tau}/A = G_1(s)e^{-s\theta} + G_2(s)[\cosh(B + Cs)^{1/2}(2\pi - \theta) - e^{-2\pi s}\cosh(B + Cs)^{1/2}\theta] \quad (12)$$

where

$$G_1(s) = -(1 + 2s\sinh\pi s/2)/2(s^2+1) \times (s^2 - Cs - B)\sinh(\pi s/2) \quad (13)$$

and

$$G_2(s) = s(1 + 2s\sinh\pi s/2)/2(s^2+1)(s^2 - Cs - B) \times (B + Cs)^{1/2}\sinh(B + Cs)^{1/2}2\pi\sinh(\pi s/2) \quad (14)$$

Eq. (12) may be put in the form

$$\bar{\tau}/A = H(\theta,s)/(s^2+1)\sinh(\pi s/2) \quad (15)$$

for which the sum of the residues

$$\sum_{j=1}^{\infty} \text{Res}\left[e^{s_j t}\frac{\bar{\tau}}{A}(\theta,s_j)\right]$$

yields the time-periodic temperature distribution.

The expression for the inverse transform of Eq. (12) can be put in a relatively compact form by introducing the following definitions:

$$a = +(1/2^{1/2})[B + (B^2 + C^2)^{1/2}]^{1/2}$$

$$b = +(1/2^{1/2})[-B + (B^2 + C^2)^{1/2}]^{1/2}$$

$$a_n = +(1/2^{1/2})[B + (B^2 + 4n^2C^2)^{1/2}]^{1/2}$$

$$b_n = +(1/2^{1/2})[-B + (B^2 + 4n^2C^2)^{1/2}]^{1/2}$$

$$R = +[a^2 + b^2]^{1/2},\ \rho = \tan^{-1}b/a$$

$$R_n = +[a_n{}^2 + b_n{}^2]^{1/2},\ \rho_n = \tan^{-1}(b_n/a_n)$$

$$K(\theta) = +[\sinh^2 a\theta + \sin^2 b\theta]^{1/2},\ \kappa(\theta) = \tan^{-1}(\coth a\theta \tan b\theta)$$

$$K_n(\theta) = +[\sinh^2 a_n\theta + \sin^2 b_n\theta]^{1/2}$$

$$\kappa_n(\theta) = \tan^{-1}(\coth a_n\theta \tan b_n\theta)$$

$$L(\theta) = +[\sinh^2 a\theta + \cos^2 b\theta]^{1/2},\ \lambda(\theta) = \tan^{-1}(\tanh a\theta \tan b\theta)$$

$$L_n(\theta) + [\sinh^2 a_n\theta + \cos^2 b_n\theta]^{1/2}$$

$$\lambda_n(\theta) = \tan^{-1}(\tanh a_n\theta \tan b_n\theta)$$

$$M = +[(1+B)^2 + C^2]^{1/2},\ \mu = \tan^{-1}C/(1+B)$$

$$M_n = +[(4n^2+B)^2 + 4n^2C^2]^{1/2},\ \mu_n = \tan^{-1}2nC/(4n^2+B)$$

$$P(\theta) = K(\pi)L(\theta)/2MRL(\pi)$$

$$p(\theta,\psi) = [\kappa(\pi) - \mu - \rho - \lambda(\pi) - \pi/2 + \lambda(\theta) + \psi]$$

$$Q(\theta) = K(\theta)/2MR,\ q(\theta,\psi) = [-\mu - \rho - \pi/2 + \kappa(\theta) + \psi]$$

$$P_n(\theta) = (-1)^n nK_n(\pi)L_n(\theta)/(4n^2-1)M_nR_nL_n(\pi)$$

$$p_n(\theta,\psi) = [\kappa_n(\pi) - \mu_n - \rho_n - \lambda_n(\pi) + \pi/2 + \lambda_n(\theta) + 2n\psi]$$

$$Q_n(\theta) = (-1)^n nK_n(\theta)/(4n^2-1)M_nR_n$$

$$q_n(\theta,\psi) = \left[-\mu_n - \rho_n + \frac{\pi}{2} + \kappa_n(\theta) + 2n\psi\right] \quad (16)$$

In terms of these definitions, the dimensionless temperature may be written:

$$\frac{\tau}{A} = P(\theta)\cos p(\theta,\psi) - Q(\theta)\cos q(\theta,\psi) + \frac{1}{2M}\cos(\mu + \theta - \psi) + \frac{4}{\pi}\sum_{n=1}^{\infty} P_n(\theta)\cos p_n(\theta,\psi) - \frac{4}{\pi}\sum_{n=1}^{\infty} Q_n(\theta)\cos q_n(\theta,\psi) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n^2-1)M_n} \times \cos(\mu_n + 2n\theta - 2n\psi) + \frac{1}{\pi B} \quad (17)$$

Temperature of Continuous Shell

The temperature distribution in a continuous cylindrical shell has been studied in detail in Ref. (1). For comparison purposes, the method outlined above will be used to obtain a solution in infinite series. Equation (9) is solved using the boundary conditions.

$$\bar{\tau}(0,s) = \bar{\tau}(2\pi,s)$$

$$d\bar{\tau}/d\theta)_{\theta=0} = d\bar{\tau}/d\theta)_{\theta=2\pi} \quad (18)$$

and the temperature transform is given by

$$\bar{\tau}/A\big|_{\substack{\text{closed}\\\text{cylinder}}} = G_1(s)e^{-s\theta} + G_3(s)[s\cosh(B + Cs)^{1/2} \times (\pi - \theta) + (B + Cs)^{1/2}\sinh(B + Cs)^{1/2}(\pi - \theta)] \quad (19)$$

where $G_1(s)$ is defined by Eq. (13) and

$$G_3(s) = \frac{(1 - e^{-2\pi s})}{2(B + Cs)^{1/2}\sinh(B + Cs)^{1/2}\pi}G_1(s) \quad (20)$$

The second term of Eq. (19) does not contribute to a periodic steady-state solution. The inverse transform is found to be

$$\frac{\tau}{A}\bigg|_{\substack{\text{closed}\\\text{cylinder}}} = \frac{1}{2M}\cos(\mu + \theta - \psi) - \frac{2}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{(4n^2-1)M_n} \times \cos(\mu_n + 2n\theta - 2n\psi) + \frac{1}{\pi B} \quad (21)$$

Of course, the value of θ in Eq. (21) can be chosen arbitrarily and the temperature profile over the entire surface is obtained by merely varying ψ.

Special Cases

Stationary discontinuous cylinder

A special case of Eq. (17) is when C is set equal to zero ($\omega \to 0$) and ψ is replaced by θ_s. This corresponds to the situation when the cylinder is stationary with the sun normal at $\theta = \theta_s$. The definitions given by Eq. (16) reduce to the following:

$$a = a_n = B^{1/2},\ b = b_n = 0 \quad (22a)$$

$$R = R_n = B^{1/2},\ \rho = \rho_n = 0 \quad (22b)$$

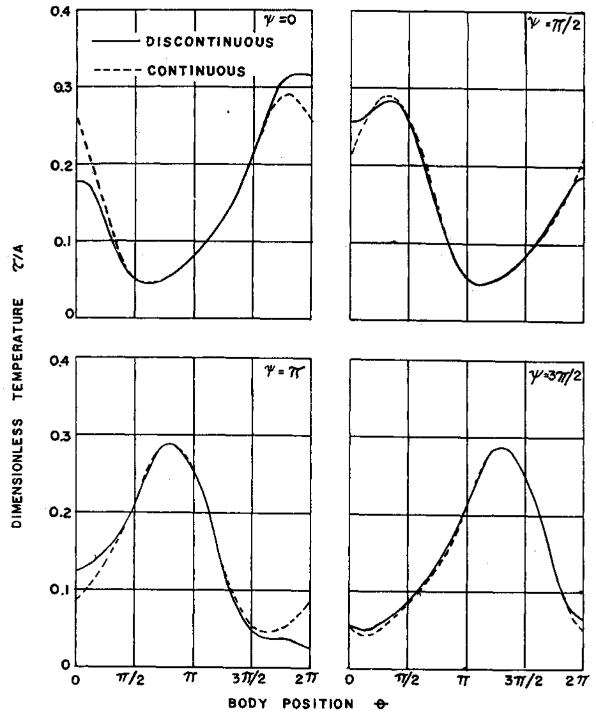

Fig. 2 Temperature distribution, ω = 8.71 rad/hr.

$$K(\theta) = K_n(\theta) = \sinh B^{1/2}\theta,\ \varkappa(\theta) = \varkappa_n(\theta) = 0 \quad (22c)$$

$$L(\theta) = L_n(\theta) = \cosh B^{1/2}\theta,\ \lambda(\theta) = \lambda_n(\theta) = 0 \quad (22d)$$

$$M = (1 + B),\ M_n = (4n^2 + B),\ \mu = \mu_n = 0 \quad (22e)$$

$$P(\theta) = \frac{\sinh B^{1/2}\pi \cosh B^{1/2}\theta}{2B^{1/2}(1 + B)\cosh B^{1/2}\pi},\ p(\theta,\theta_s) = \theta_s - \frac{\pi}{2} \quad (22f)$$

$$Q(\theta) = \frac{\sinh B^{1/2}\theta}{2B^{1/2}(1 + B)},\ q(\theta,\theta_s) = \theta_s - \frac{\pi}{2} \quad (22g)$$

$$P_n(\theta) = \frac{(-1)^n n \sinh B^{1/2}\pi \cosh B^{1/2}\theta}{B^{1/2}(4n^2 - 1)(4n^2 + B)\cosh B^{1/2}\pi},\quad p_n(\theta,\theta_s) = 2n\theta_s + \frac{\pi}{2} \quad (22h)$$

$$Q_n(\theta) = \frac{(-1)^n n \sinh B^{1/2}\theta}{B^{1/2}(4n^2 - 1)(4n^2 + B)},\ q_n(\theta,\theta_s) = 2n\theta_s + \frac{\pi}{2} \quad (22i)$$

Substituting these expressions into Eq. (17), the following equation is obtained.

$$\left.\frac{\tau}{A}\right|_{\text{stationary}} = \frac{1}{2(B + 1)}\cos(\theta - \theta_s) - \frac{1}{8\pi}\sum_1^{\infty}\frac{(-1)^n\cos 2n(\theta - \theta_s)}{(n^2 - 1/4)(n^2 + B/4)} + G(\theta_s) \times [\cosh B^{1/2}\theta - \cosh B^{1/2}(2\pi - \theta)] + \frac{1}{\pi B} \quad (23)$$

where

$$G(\theta_s) = \frac{1}{B^{1/2}\sinh(B^{1/2}2\pi)} \times \left[\frac{1}{4\pi}\sum_1^{\infty}\frac{(-1)^n n\sin 2n\theta_s}{(n^2 - 1/4)(n^2 + B/4)} - \frac{\sin\theta_s}{2(B + 1)}\right] \quad (24)$$

This result is consistent with the analysis in Ref. 4.

The analogous case for a stationary continuous cylinder is found by taking C and ψ as zero in Eq. (17) or Eq. (21). Either of these equations yields

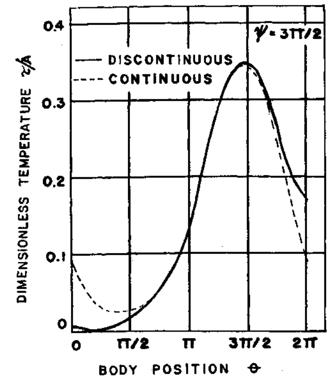

Fig. 3 Temperature distribution, ω = 1.386 rad/hr, $\psi = 3\pi/2$.

$$\left.\frac{\tau}{A}\right|_{\substack{\text{stationary}\\ \text{closed cylinder}}} = \frac{1}{2(B + 1)}\cos\theta - \frac{1}{8\pi}\sum_1^{\infty}\frac{(-1)^n\cos 2n\theta}{(n^2 - 1/4)(n^2 + B/4)} + \frac{1}{\pi B} \quad (25)$$

Fast moving cylinders

If C is taken very large ($\omega \to \infty$), the first six terms in Eq. (17) become negligible in comparison to $1/\pi B$. Thus

$$\tau_{\text{high speed}} \sim A/\pi B \quad (26)$$

From the definitions of A and B and T_m,

$$\tau_{\text{high speed}} \sim \tfrac{1}{4} \quad (27)$$

This is equivalent to saying that

$$T_{\text{high speed}} \sim T_m \quad (28)$$

which, in view of Eq. (5), is in accordance with intuition. The same conclusion is obtained when Eq. (21) is considered under the same conditions.

Numerical Example

Consider a discontinuous cylinder having the properties given in the example of Ref. 1, namely, $\alpha = \epsilon = 1$; $r = 1$ ft; $t = 1/200$ ft; $k = 100$ Btu/ft-hr-°R; $k/\rho c = 3$ ft²/hr; $S = 442$ Btu/ft²-hr; and $\omega = 8.71$ rad/hr.

Equation (17) was solved numerically, and the results are shown in Fig. 2 for $\psi = 0$, $\pi/2$, π and $3\pi/2$. Results are compared to those for a similar cylinder of continuous cross-section using Eq. (21). It is noted that, whereas the shape of the temperature profile for the closed cylinder remains unchanged when shifted by the rotational displacement, the temperature profile of a discontinuous cylinder varies during a revolution. The greatest changes occur near the discontinuity. Five terms were taken in the series summations of Eqs. (17) and (21). The results for the closed cylinder are very close to those obtained by the closed-form solution in Ref. 1. The difference between the continuous and discontinuous cylinder is clearly observed in Fig. 3, which shows the temperature profiles when $\omega = 1.386$ rad/hour and at $\psi = 3\pi/2$.

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